Quantum Noise and Fluctuations in Gravitation and Cosmology

B. L. Hu[a*], Albert Roura[a], Sukanya Sinha[b], E. Verdaguer[c]
[a]Department of Physics, University of Maryland, College Park, Maryland 20742-4111, U.S.A.
[b]Indian Statistical Institute, Bangalore Centre, 8th Mile, Mysore Road, Bangalore-560059, INDIA
[c]Departament de Fisica Fonamental and C.E.R. in Astrophysics, Particles and Cosmology, Universitat de Barcelona, Av. Diagonal 647, 08028 Barcelona, Spain
((UMD Physics Preprint number: umdpp 03-045, April 12, 2003))
Invited Talk given by BLH at the First International Symposium on Fluctuations and Noise Sponsored by SPIE, 1-4 June 2003, Santa Fe, New Mexico. Paper number 5111-46

We give a short update of our research program on nonequilibrium statistical field theory applied to quantum processes in the early universe and black holes, as well as the development of stochastic gravity theory as an extension of semiclassical gravity and an intermediary in the 'bottom-up' approach to quantum gravity.

PROLOGUE

(BLH) "It is no secret that the message of this conference is 'NOISE IS GOOD'. In this talk I want to show that *not only is noise good, it is absolutely essential...* " . These words were not uttered in this conference, as it would have been acausal, but ten years ago, at a workshop devoted to the nascent yet fascinating subject of Fluctuations and Order [1]. After a decade, the tenor and objectives of this statement are even closer to reality, as witnessed by the strong focus and great variety of this symposium.

I substantiated this claim then by an enumeration of the many processes in gravitation and cosmology where quantum noise and fluctuations play an active role:
1. Particle creation as parametric amplification of vacuum fluctuations
2. Thermal radiance from accelerated observers and black holes as fluctuation-dissipation phenomena
3. Entropy generation from quantum stochastic and kinetic processes
4. Phase transitions in the early universe as noise-induced processes
5. Galaxy formation from primordial quantum fluctuations
6. Anisotropy dissipation from particle creation as backreaction processes
7. Dissipation in quantum cosmology and the issue of the initial state
8. Decoherence, backreaction and the semiclassical limit of quantum gravity
9. Stochastic spacetime and continuum limit, gravity as an effective theory
10. Topology change in spacetime and loss of quantum coherence problems
11. Gravitational entropy, singularity and time asymmetry
12. 'Birth' of the universe as a spacetime fluctuation and tunneling phenomenon

The above list was prepared for cosmological issues. Further developments in the last ten years focusing on the effects of noise and fluctuations using the statistical field theory approach we have developed (for a review, see [2,3]) include, related to Topic 1 : Preheating in post-inflationary cosmology [4,5]; Topic 2: Thermal and near-thermal radiance in detectors, moving mirrors, black holes and cosmology [6–9]; Topic 3: Correlation entropy [10,11]; Topic 4: Defect formation [12], tunneling induced by quantum and thermal noise fluctuations [13,14]. Topic 5 is presently under pursuit [16] and partially summarized in the last section of this paper. Topic 9 includes preliminary work on wave propagation in stochastic spacetimes [17] and mesoscopic fluctuations [18].

In black hole physics, focusing again on fluctuations, one can mention topics on black hole fluctuations and backreaction [19,20] (see references therein for other related work) and the energetics and dynamics of black

*Corresponding author. Electronic address: hub@physics.umd.edu

hole phase transition [21–23]. The wish list would include: applications of stochastic gravity to the statistical mechanical definition of entropy and statistical field theory of black hole nonequilibrium thermodynamics.

Using statistical quantum field theory , Topics 1, 2, 3, 6 are quite well understood, 7, 8 only partially, and Topics 4, 5, 9 are currently under pursuit. Certain concepts in Topics 10-12 may not even be well-defined (e.g., what does the 'birth' of the universe mean?), but the newly established theory of stochastic gravity (for a review, see [24–27], an ongoing development, see [28]) may offer alternative ways to address these issues, as well as provide an intermediary towards quantum gravity (our definition is different from the ordinary, see [29]).

In this talk I will focus on the theory of noise in quantum fields and how quantum fluctuations could have played an active and even decisive role in many fundamental processes in cosmology and gravitation, especially near the Planck time (10^{-43} sec from the Big Bang). I will describe how stochastic gravity theory can be understood easily from an open system conceptual framework, and give two examples of its applications: fluctuations and backreaction in black holes, and structure formation in the early universe.

The Planck time is the time when many familiar features of spacetime depicted by Einstein's theory of general relativity give way to an as-yet-unknown quantum theory of gravity depicting the microstructure of spacetime. (Some theoreticians believe the superstring theory is the answer.) Just below the Planck energy we believe the universe can be adequately described by a semiclassical theory of gravity [30–32], where quantized matter fields coexist with a classical spacetime. Many qualitative changes are believed to have taken place at this energy scale, amongst them the formation of spacetime depictable as a manifold, the emergence of time, the creation of particle pairs from the vacuum, the growth of fluctuations as seeds for galaxies, and possible phase transitions and the ensuing entropy generation processes. It is also the cross-over point of quantum to classical and micro to macroscopic transitions.[1]

In Section I, we first explain the origin and nature of noise in quantum systems interacting with an environment, using the influence functional method. (For non-Ohmic bath at low temperatures, colored noise and nonlocal dissipation would appear; and for nonlinear coupling multiplicative noise is generally expected. A generalized fluctuation-dissipation relation for these systems can be proven, and the stochastic (master, Langevin and Fokker-Planck) equations derived, depicting the dissipative dynamics of the open system under the influence of noise. We then discuss stochastic gravity theory in Sec. 2 and the two examples in Sections 3 and 4.

I. QUANTUM FLUCTUATIONS IN OPEN SYSTEMS

We begin by describing how the concepts of quantum open systems can be of use in the treatment of statistical mechanical problems involving quantum fields. We start with two subsystems A and B. When the precise information of subsystem B is not required, but only its averaged effect on subsystem A is of interest, one can *coarse-grain* B and include its averaged effect on A, which involves finding the *backreaction* on A. In so doing A is rendered an open system, with B acting as its environment. For the analysis of open systems with backreaction from the environments the influence functional (IF) formalism of Feynman and Vernon [35,36] proves useful. Let us first use a simple example from quantum mechanics to illustrate the idea and the method. We then show how we can use this framework to address issues in gravitation and cosmology.

---

[1]It is for this reason that I think one can view *general relativity as geometro-hydrodynamics* [34], *semiclassical gravity as a mesoscopic physics* [33], and take a *kinetic theory approach to quantum gravity* [29]. The only difference is that instead of dealing with the quantum to classical and micro- to macro- transition in the state of matter and fields we are dealing with the corresponding issues for spacetime and geometry.

A. Stochastic Effective Action in Quantum Open Systems

We begin with a brief schematic summary of the IF formalism as applied to a simple system. Consider a system S, described by the degrees of freedom x, interacting with an environment E, described by the degrees of freedom q.[2] The full closed quantum system $S+E$ is described by a density matrix $\rho(x,q;x',q',t)$. If we are interested only in the state of the system as influenced by the overall effect, but not the precise state of the environment, i.e, the dynamics of the open system, then the reduced density matrix $\rho_r(x,x',t) = \int dq\ \rho(x,q;x',q,t)$ would provide the relevant information. (The subscript r stands for reduced.) Assuming that the action of the coupled system decomposes as $S = S_s[x]+S_e[q]+S_{int}[x,q]$, and that the initial density matrix factorizes (i.e., takes the tensor product form), $\rho(x,q;x',q',t_i) = \rho_s(x,x',t_i)\rho_e(q,q',t_i)$, the reduced density matrix is given by

$$\rho_r(x,x',t) = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx'_i \ \int_{x_i}^{x_f} Dx \ \int_{x'_i}^{x'_f} Dx' \ e^{i(S_s[x]-S_s[x']+S_{IF}[x,x',t])} \ \rho_r(x_i,x'_i,t_i\ ) \tag{I.1}$$

where S_{IF} is the influence action related to the influence functional $\mathcal{F}$ defined by

$$\mathcal{F}[x,x'] \equiv e^{iS_{IF}[x,x',t]} \equiv \int \ dq_f \ dq_i \ dq'_i \ \int_{q_i}^{q_f} Dq \ \int_{q'_i}^{q_f} Dq' \ e^{i(S_e[q]+S_{int}[x,q]-S_e[q']-S_{int}[x',q'])}\rho_e(q_i,q'_i,t_i). \tag{I.2}$$

S_{IF} in general is complex. Retaining only quadratic terms (an approximation which covers many of the interesting applications that we will consider later), we may write

$$S_{IF}(x,x') = \int \ dt \ dt' \ \{\frac{1}{2}(x-x')(t)D(t,t')(x+x')(t') + \frac{i}{2}(x-x')(t)N(t,t')(x-x')(t')\} \tag{I.3}$$

where D and N stand for the real dissipation and noise kernels respectively. Note that in this quadratic order approximation, the influence action $S_{IF}(x,x')$ is related to the closed-time-path(CTP) or in-in effective action (for details on the CTP effective action see [38]) $\Gamma_{CTP}[x,x']$ through

$$\Gamma_{CTP}[x,x'] = S[x] - S[x'] + S_{IF}[x,x']. \tag{I.4}$$

The equation of motion obtained from the CTP effective action for the expectation values is clearly seen to be real and causal [38]. It reads

$$\frac{\delta}{\delta x(t)}\Gamma_{CTP}[x,x']\bigg|_{x'=x=\bar{x}} = 0 \tag{I.5}$$

From the influence functional a Langevin equation for the system dynamics may be derived by a formal procedure, first introduced by Feynman and Vernon [35], which consists of introducing a Gaussian stochastic source $\xi(t)$ with $\langle\xi(t)\rangle_\xi = 0$ and $\langle\xi(t)\xi(t')\langle= N(t,t')$ and defining an improved or stochastic effective action as

$$S_{eff}[x,x';\xi] = S_s[x] - S_s[x'] + \mathcal{R}S_{IF}[x,x'] + \xi(x-x') \tag{I.6}$$

such that $\left\langle e^{iS_{eff}[x,x';\xi]}\right\rangle_\xi = e^{i\Gamma_{CTP}[x,x']}$. This leads to equations of motion with a stochastic force:

$$\frac{\delta S_{eff}[x,x';\xi]}{\delta x}\bigg|_{x=x'} = 0 \quad \text{or, equivalently,} \quad \frac{\delta\Gamma_{\rm CTP}[\rm x,x';\xi]}{\delta \rm x}\bigg|_{\rm x=x'} = 0. \tag{I.7}$$

---

[2] We are labeling the degrees of freedom of the system and the environment by single letters x and q with the understanding that they can represent multiple or even infinite degrees of freedom, e.g. corresponding to a field [37].

The equation of motion obtained from (I.7) using (I.6) is

$$\frac{\partial S_s}{\partial x(t)} + \int dt' \; \gamma(t,t') \frac{dx(t')}{dt'} = \xi \tag{I.8}$$

where $D(t,t') = -\partial_{t'}\gamma(t,t')$. Being now in the form of a Langevin equation, the physical meaning of the γ and N kernels in Eq. (I.8) becomes clearer. Both the terms involving γ and ξ represent the backreaction of the environment on the system. However, γ (or more properly the odd part of γ) is associated with dissipation and ξ is a stochastic noise term associated with random fluctuations of the system, exactly as the terms are interpreted in the context of Brownian motion. Averaging (I.8) over the noise using the appropriate probability distribution will give the semiclassical equation of motion for the mean value of x. Since the noise and dissipation arise by considering a subsystem within a closed system (as is done here, as opposed to being put in by hand), they are in general related by a set of generalized fluctuation-dissipation relations (FDR), which can be represented by a linear, non-local relation of the following form, provided that the Hamiltonian for the environment and the system-environment interaction are time independent and the initial state of the environment is stationary with respect to the Hamiltonian of the environment:

$$N(t-t') = \int d(s-s') K(t-t', s-s') \gamma(s-s') \tag{I.9}$$

To keep the discussion simple, we have written the noise and dissipation kernels in terms of single scalar functions. However, the method is general enough to encompass multiple noise and dissipation kernels and cases where the kernels are tensorial, as in the stochastic gravity theory discussed later.

II. STOCHASTIC GRAVITY AND METRIC FLUCTUATIONS

Stochastic semiclassical gravity [39] of the 90's is a theory naturally evolved from semiclassical gravity [32] of the 80's and quantum field theory in curved spacetimes [30,31] of the 70's. Whereas semiclassical gravity is based on the semiclassical Einstein equation with sources given by the expectation value of the stress-energy tensor of quantum fields, stochastic semiclassical gravity is based on the Einstein-Langevin equation, which has in addition stochastic sources with correlation functions characterized by the noise kernel. the noise kernel. The noise kernel is the vacuum expectation value of the (operator-valued) stress-energy bi-tensor which describes the fluctuations of quantum matter fields in curved spacetimes.

A. From Semiclassical to Stochastic Gravity

The first stage in the road to stochastic gravity begins with *quantum field theory in curved spacetime*, which describes the behavior of quantum matter fields propagating in a specified (not dynamically determined by the quantum matter field as a source) background gravitational field. For a scalar field ϕ it obeys the wave equation $(\Box + m^2)\phi(x) = 0$ where $\Box$ is the Laplace-Beltrami operator, which contains the imprint of the curvature of the background spacetime. In this theory the gravitational field is given by the classical spacetime metric determined from classical sources by the classical Einstein equations, and the quantum fields propagate as test fields in such a spacetime. For time dependent spacetime geometry it may not be possible to define a physically meaningful vacuum state for the quantum field at all times. Assuming that one defines a vacuum state at some initial time, the vacuum state at a latter time will differ from that defined initially because particles are created in the intervening time. An important process described by quantum field theory in curved spacetime is indeed particle creation from the vacuum (and effects of vacuum fluctuations and polarizations) in the early universe [41] and Hawking radiation in black holes [41].

The second stage in the description of the interaction of gravity with quantum fields is *back-reaction*, i.e., the effect of quantum fields on the spacetime geometry. The dynamic classical spacetime metric creates particles of the quantum field and these in turn provide a backreaction on the spacetime metric which alters its dynamics in response. One assumes a general class of spacetime where the quantum fields live in and act

on, and seek a solution which satisfies simultaneously the Einstein equation for the spacetime and the field equations for the quantum fields. The Einstein equation which has the expectation value of the stress-energy operator of the quantum matter field as the source is known as the *semiclassical Einstein equation*:

$$G_{\mu\nu}(g_{\alpha\beta}) = \kappa \langle \hat{T}_{\mu\nu} \rangle_q \tag{II.1}$$

where $\hat{T}_{\mu\nu}$ is the stress-energy tensor operator of, say, a free scalar field ϕ, $G_{\mu\nu}$ is the Einstein tensor, $\kappa = 8\pi G_N$ and G_N is Newton's constant. Here $\langle\,\rangle_q$ denotes the expectation value taken with respect to some quantum state compatible with the symmetries of the background spacetime, a classical object. The theory obtained from a self-consistent solution of the geometry of the spacetime and the quantum field is known as *semiclassical gravity*. Incorporating the backreaction of the quantum matter field on the spacetime is thus the central task in semiclassical gravity.

Studies of the semiclassical Einstein equation for the backreaction problems have been carried out in the last two decades by many authors for cosmological and black hole spacetimes. A well-known example of semiclassical gravity is the damping of anisotropy in Bianchi universes (which is the basis of chaotic cosmology in the 70's) by the backreaction of vacuum particle creation, and inflationary cosmology [42–44] of the 80's driven by a constant vacuum energy density source such as the expectation value of a Higgs field.

In analogy with the open system dynamics described in Section I A, Eq.(II.1) is equivalent to Eq.(I.5) where the degrees of freedom x of the system are identified with the metric $g_{\alpha\beta}$ and those of the environment q are identified with the scalar field $\phi(x)$. However, from the discussion in the last section it is also clear that Eq. (I.5) , and hence also the semiclassical Einstein Eq. (II.1) results on averaging the full Langevin-type Eq. (I.7) over noise. Thus the semiclassical Einstein equation incorporates the dissipation but misses out the fluctuation aspect of the backreaction. The recognition of this crucial point [45] ushered in a new theory known as *stochastic semiclassical gravity*, (or in short, stochastic gravity, as there is no confusion in this context as to where the stochasticity originates). Aided by the concept of open systems and the techniques of the influence functional and the Closed Time Path (CTP) effective action, stochastic gravity is the new framework for the consideration of backreaction because it encompasses fluctuations and dissipation (from particle creation and other quantum field processes) on the same footing. Spacetime dynamics is now governed by a stochastic generalization of the semiclassical Einstein equation known as the Einstein-Langevin equation, the analog of Eq. (I.8) in the context of semiclassical gravity (SCG). Schematically the Einstein-Langevin equation takes on the form

$$\tilde{G}_{\mu\nu}(x) = \kappa \left(T^{\rm c}_{\mu\nu} + T^{\rm qs}_{\mu\nu} \right), \quad T^{\rm qs}_{\mu\nu} \equiv \langle \hat{T}_{\mu\nu} \rangle_{\rm q} + T^{\rm s}_{\mu\nu} \tag{II.2}$$

Here, $T^{\rm c}_{\mu\nu}$ is due to classical matter or fields, $\langle \hat{T}_{\mu\nu} \rangle_q$ is the expectation value of the stress tensor of the quantum field, and $T^{\rm qs}_{\mu\nu}$ is a new stochastic term which is related to the quantum fluctuations of $T_{\mu\nu}$ for the state of the field under consideration. Taking the average of (II.2) with respect to the noise distribution will lead to the conventional semiclassical Einstein equation. It is in this context that SCG is regarded as a mean field theory.

The fundamentals of this new theory were developed via two approaches: the axiomatic and the functional. The axiomatic approach is useful to see the structure of the theory from the framework of semiclassical gravity, showing the link from the mean value of the stress-energy tensor to its correlation functions. The functional approach uses the Feynman-Vernon influence functional and the Schwinger-Keldysh closed-time-path effective action methods which are convenient for computations. It also brings out the open systems concepts and the statistical and stochastic contents of the theory such as dissipation, fluctuations, noise and decoherence. There was also theoretical work on the properties of the stress energy bi-tensor and its vacuum expectation value, the noise kernel. See, e.g., [46–52]. For a broader exploration of ideas and issues based on this theory read the reviews [24,27]; For a pedagogical introduction with applications, see [26,25].

Thus with the aid of the open system viewpoint it is easy to see that stochastic gravity is a natural extension of the well-established semiclassical gravity theory and a useful framework for the considerations of fluctuations in quantum matter fields and dissipative dynamics of classical spacetimes, including metric fluctuations. Stochastic gravity can address many important issues related to nonequilibrium quantum field processes in curved spacetimes and find applications to many problems in gravitation and cosmology.

B. Stochastic Gravity in relation to Quantum Gravity

Before embarking on the discussion of some applications of stochastic gravity, let us illustrate the theory with a simple toy model which minimizes the technical complications. The model will be useful to clarify the role of the noise kernel and illustrate the relationship between the semiclassical, stochastic and quantum descriptions. Let us assume that the gravitational equations are described by a linear field $h(x)$ whose source is a massless scalar field $\phi(x)$ which satisfies the Klein-Gordon equation in flat spacetime $\Box\phi(x) = 0$. The field stress-energy tensor is quadratic in the field, and independent of $h(x)$. The classical gravitational field equations will be given by[3]

$$\Box h(x) = \kappa T(x), \tag{II.3}$$

where $T(x)$ is the (scalar) trace of the stress-energy tensor, $T(x) = \partial_a\phi(x)\partial^a\phi(x)$ and $\kappa \equiv 16\pi G$, where G is Newton's constant. Note that this is not a self-consistent theory since $\phi(x)$ does not react to the gravitational field $h(x)$. We should also emphasize that this model is not the standard linearized theory of gravity in which T is also linear in $h(x)$. It captures, however, some of the key features of linearized gravity.

In the Heisenberg representation the quantum field $\hat{h}(x)$ satisfies

$$\Box\hat{h}(x) = \kappa\hat{T}(x). \tag{II.4}$$

Since $\hat{T}(x)$ is quadratic in the field operator $\hat{\phi}(x)$ some regularization procedure has to be assumed in order for (II.4) to make sense. Since we work in flat spacetime we may simply use a normal ordering prescription to regularize the operator $\hat{T}(x)$. The solutions of this equation, i.e. the field operator at the point x, $\hat{h}(x)$, may be written in terms of the retarded propagator $G(x, y)$ as,

$$\hat{h}(x) = \hat{h}^{(0)}(x) + \kappa\int dx' G(x, x')\hat{T}(x'), \tag{II.5}$$

where $\hat{h}^{(0)}(x)$ is the free field which carries information on the initial conditions and the state of the field. From this solution we may compute, for instance, the symmetric two point quantum correlation function (the anticommutator)

$$\frac{1}{2}\langle\{\hat{h}(x), \hat{h}(y)\}\rangle = \frac{1}{2}\langle\{\hat{h}^{(0)}(x), \hat{h}^{(0)}(y)\}\rangle + \frac{\kappa^2}{2}\int\int dx'dy' G(x, x')G(y, y')\langle\{\hat{T}(x'), \hat{T}(y')\}\rangle, \tag{II.6}$$

where the expectation value is taken with respect to the quantum state in which both fields $\phi(x)$ and $h(x)$ are quantized. (We assume for the free field, $\langle\hat{h}^{(0)}\rangle = 0$.)

We can now consider the semiclassical theory for this problem. If we assume that $h(x)$ is classical and the matter field is quantum the semiclassical theory may just be described by substituting into the classical equation (II.3) the stress-energy trace by the expectation value of the stress-energy trace operator $\langle\hat{T}(x)\rangle$, in some quantum state of the field $\hat{\phi}(x)$. Since in our model $\hat{T}(x)$ is independent of $h(x)$ we may simply renormalize its expectation value using normal ordering, then for the vacuum state of the field $\hat{\phi}(x)$, we would simply have $\langle\hat{T}(x)\rangle_0 = 0$. The semiclassical theory thus reduces to

$$\Box h(x) = \kappa\langle\hat{T}(x)\rangle. \tag{II.7}$$

The two point function $h(x)h(y)$ that one may derive from this equation depends on the two point function $\langle\hat{T}(x)\rangle\langle\hat{T}(y)\rangle$ and clearly cannot reproduce the quantum result (II.6) which depends on the expectation value

[3]In this article we use the $(+, +, +)$ sign conventions of Ref. [53], and units in which $c = \hbar = 1$.

of the two point operator $\langle\{\hat{T}(x),\hat{T}(y)\}\rangle$. That is, the semiclassical theory entirely misses the fluctuations of the stress-energy operator $\hat{T}(x)$.

Let us now see how we can extend the semiclassical theory in order to account for such fluctuations. The first step is to characterize these fluctuations. For this, we introduce the noise kernel as the physical observable that measures the fluctuations of the stress-energy operator $\hat{T}$. Define

$$N(x,y)=\frac{1}{2}\langle\{\hat{t}(x),\hat{t}(y)\}\rangle \tag{II.8}$$

where $\hat{t}(x)=\hat{T}(x)-\langle\hat{T}(x)\rangle$. The bi-scalar $N(x,y)$ is real and positive-semidefinite, a consequence of $\hat{t}$ being self-adjoint. A simple proof can be given as follows. Let $|\psi\rangle$ be a given quantum state and let $\hat{Q}$ be a self-adjoint operator, $\hat{Q}^{\dagger}=\hat{Q}$, then one can write $\langle\psi|\hat{Q}\hat{Q}|\psi\rangle=\langle\psi|\hat{Q}^{\dagger}Q|\psi\rangle=|\hat{Q}|\psi\rangle|^2\geq 0$. Now let $\hat{t}(x)$ be a self-adjoint operator, then if we define $\hat{Q}=\int dx f(x)\hat{t}(x)$ for an arbitrary well behaved function $f(x)$, the previous inequality can be written as $\int dxdy f(x)\langle\psi|\hat{t}(x)\hat{t}(y)|\psi\rangle f(y)\geq 0$, which is the condition for the noise kernel to be positive semi-definite. Note that when considering the inverse kernel $N^{-1}(x,y)$, it is implicitly assumed that one is working in the subspace obtained from the eigenvectors which have strictly positive eigenvalues when the noise kernel is diagonalized.

By the positive semi-definite property of the noise kernel $N(x,y)$ it is possible to introduce a Gaussian stochastic field as follows:

$$\langle\xi(x)\rangle_s=0,\quad \langle\xi(x)\xi(y)\rangle_s=N(x,y). \tag{II.9}$$

where the subscript s means a statistical average. These equations entirely define the stochastic process $\xi(x)$ since we have assumed that it is Gaussian. Of course, higher correlations could also be introduced but we just try to capture the fluctuations to lowest order.

The extension of the semiclassical equation may be simply performed by adding to the right-hand side of the semiclassical equation (II.7) this stochastic source $\xi(x)$ which accounts for the fluctuations of $\hat{T}$ as follows,

$$\Box h(x)=\kappa\left(\langle\hat{T}(x)\rangle+\xi(x)\right). \tag{II.10}$$

This equation is in the form of a Langevin equation: the field $h(x)$ is classical but stochastic and the observables we may obtain from it are correlation functions for $h(x)$. In fact, the solution of this equation may be written in terms of the retarded propagator as,

$$h(x)=h^{(0)}(x)+\kappa\int dx' G(x,x')\left(\langle\hat{T}(x')\rangle+\xi(x')\right), \tag{II.11}$$

from where the two point correlation function for the classical field $h(x)$, after using the definition of $\xi(x)$ and that $\langle h^{(0)}(x)\rangle_s=0$, is given by

$$\langle h(x)h(y)\rangle_s=\langle h^{(0)}(x)h^{(0)}(y)\rangle_s+\frac{\kappa^2}{2}\int\int dx'dy' G(x,x')G(y,y')\langle\{\hat{T}(x'),\hat{T}(y')\}\rangle. \tag{II.12}$$

Note that in writing $\langle\ldots\rangle_s$ here we are assuming a double stochastic average, one is related to the stochastic field $\xi(x)$ and the other is related to the free field $h^{(0)}(x)$ which is assumed also to be stochastic with a distribution function to be specified.

Comparing (II.6) with (II.12) we see that the respective second term on the right-hand side are identical provided the expectation values are computed in the same quantum state for the field $\hat{\phi}(x)$ (recall that we have assumed $T(x)$ does not depend on $h(x)$). The fact that the field $h(x)$ is also quantized in (II.6) does not change the previous statement. The nature of the first term on the right-hand side of Eqs. (II.6) and (II.12) is different: in the first case it is the two point quantum expectation value of the free quantum field $\hat{h}^{(0)}$ whereas in the second case it is the stochastic average of the two point classical homogeneous field $h^{(0)}$,

which depends on the initial conditions. Now we can still make these terms equal to each other if we assume for the homogeneous field $h^{(0)}$ a Gaussian distribution of initial conditions such that

$$\langle h^{(0)}(x)h^{(0)}(y)\rangle_s = \frac{1}{2}\langle\{\hat{h}^{(0)}(x), \hat{h}^{(0)}(y)\}\rangle. \tag{II.13}$$

This Gaussian stochastic field $h^{(0)}(x)$ can always be defined due to the positivity of the anti-commutator. Thus, under this assumption on the initial conditions for the field $h(x)$ the two point correlation function of (II.12) equals the quantum expectation value of (II.6) exactly. An interesting feature of the stochastic description is that the quantum anticommutator of (II.6) can be written as the right-hand side of equation (II.12), where the first term contains all the information on the initial conditions for the stochastic field $h(x)$ and the second term codifies all the information on the quantum correlations of the source. This separation is also seen in the description of some quantum Brownian motion models which are typically used as paradigms of open quantum systems [13,14].

It is interesting to note that in the standard linearized theory of gravity $T(x)$ depends also on $h(x)$, both explicitly and also implicitly through the coupling of $\phi(x)$ with $h(x)$. The equations are not so simple but it is still true that the corresponding Langevin equation leads to the correct symmetrized two point quantum correlations for the metric perturbations [48,28]. Thus in a linear theory as in the model just described one may just use the statistical description given by (II.10) to compute the symmetric quantum two point function of equation (II.5). This does not mean that we can recover all quantum correlation functions with the stochastic description, see Ref. [13] for a general discussion about this point. Note that, for instance, the commutator of the classical stochastic field $h(x)$ is obviously zero, but the commutator of the quantum field $\hat{h}(x)$ is not zero for timelike separated points; this is the prize we pay for the introduction of the classical field $\xi(x)$ to describe the quantum fluctuations. Furthermore, the statistical description is not able to account for the graviton-graviton effects which go beyond the linear approximation in $\hat{h}(x)$.

III. BLACK HOLE FLUCTUATIONS AND BACKREACTION

As the first example we consider the backreaction of Hawking radiation on black holes [54] with fluctuations, i.e., how a quantum field and its fluctuations influence the behavior of the background spacetime. We will only sketch the strategy of this program based on stochastic gravity, as detailed calculations are still in progress. The formalism described in Section I A will be useful. Here we study the simpler class of problems of a quasi-static black hole in quasi-equilibrium (a box is required) with its Hawking radiation described by a scalar field. The goal is to obtain a stochastic influence action analogous to (I.6) for this model of a black hole coupled to a scalar field. From it one can derive an Einstein-Langevin equation analogous to (I.8).

We consider the simplest model of this class described by a perturbed Schwarzschild metric, used by York [55] to analyze black hole backreaction. We focus on the new aspects of noise and fluctuations, their origin and attributes.

In this model the black hole spacetime is described by a spherically symmetric static nonvacuous metric with line element of the following general form written in advanced time Eddington-Finkelstein coordinates

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\psi}\left(1 - \frac{2m}{r}\right)dv^2 + 2e^{2\psi}dvdr + r^2\, d\Omega^2 \tag{III.1}$$

where $\psi = \psi(r)$ and $m = m(r)$, $v = t + r + 2Mln\left(\frac{r}{2M} - 1\right)$ and $d\Omega^2$ is the line element on the two sphere. Hawking radiation is described by a massless, conformally coupled quantum scalar field ϕ with the classical action

$$S_m[\phi, g_{\mu\nu}] = -\frac{1}{2}\int d^n x\sqrt{-g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \xi(n)R\phi^2] \tag{III.2}$$

where $\xi(n) = \frac{(n-2)}{4(n-1)}$ (n is the dimension of spacetime) and R is the curvature scalar of the spacetime it lives in.

Let us consider linear perturbations $h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu}$ off a background Schwarzschild metric $g^{(0)}_{\mu\nu}$ with line element

$$(ds^2)^0 = \left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \tag{III.3}$$

We look for this class of perturbed metrics in the form given by (III.1), (thus restricting our consideration only to spherically symmetric perturbations):

$$e^{\psi} \simeq 1 + \epsilon\rho(r), \quad m \simeq M[1 + \epsilon\mu(r)] \tag{III.4}$$

where $\frac{\epsilon}{\lambda M^2} = \frac{1}{3} a T_H^4; a = \frac{\pi^2}{30}; \lambda = 90(8^4)\pi^2$. T_H is the Hawking temperature. (This particular parametrization of the perturbation is chosen following York's [55] notation.) Thus the only non-zero components of $h_{\mu\nu}$ are

$$h_{vv} = -\left((1 - \frac{2M}{r})2\epsilon\rho(r) + \frac{2M\epsilon\mu(r)}{r}\right), \quad h_{vr} = \epsilon\rho(r) \tag{III.5}$$

So this represents a metric with small static and radial perturbations about a Schwarzschild black hole. The initial quantum state of the scalar field is taken to be the Hartle-Hawking vacuum, which is essentially a thermal state at the Hawking temperature as far as static observers are concerned [4] and it represents a black hole in (unstable) thermal equilibrium with its own Hawking radiation.

The metric perturbation expansion induces a decomposition of the Einstein tensor $G_{\mu\nu} \simeq G^{(0)}_{\mu\nu} + \delta G_{\mu\nu}$ where $G^{(0)}_{\mu\nu}$ is the Einstein tensor for the background spacetime. The zeroth order solution gives a background metric in empty space, i.e, the Schwarzschild metric. $\delta G_{\mu\nu}$ is the linear correction to the Einstein tensor in the perturbed metric. The semiclassical Einstein equation in this approximation therefore reduces to

$$\delta G_{\mu\nu}(g^{(0)}, h) = \kappa \langle T_{\mu\nu} \rangle$$

York solved this equation to first order by using the expectation value of the energy momentum tensor for a conformally coupled scalar field in the Hartle-Hawking vacuum in the unperturbed (Schwarzschild) spacetime on the right hand side and $\delta G_{\mu\nu}$ on the left hand side is calculated using (III.5). This yields the corrections to the background metric induced by the backreaction encoded in the functions $\mu(r)$ and $\rho(r)$, which amounts to the noise-averaged backreaction effects. We are interested in the fluctuations and its effects.

We now derive the CTP effective action for this model, following the treatment of Ref. [56]. Using the metric (III.3) (and neglecting the surface terms that appear in an integration by parts) we have the action for the scalar field written perturbatively as

$$S_m[\phi, h_{\mu\nu}] \;=\; \frac{1}{2} \int d^n x \sqrt{-g^{(0)}}\; \phi \left[\Box + V^{(1)} + V^{(2)} + \cdots\right] \phi, \tag{III.6}$$

where the first and second order perturbative operators $V^{(1)}$ and $V^{(2)}$ are given by

$$\begin{aligned} V^{(1)} \;&\equiv\; -\frac{1}{\sqrt{-g^{(0)}}} \left\{ [\partial_\mu \left(\sqrt{-g^{(0)}} \bar{h}^{\mu\nu}(x)\right)]\partial_\nu + \bar{h}^{\mu\nu}(x)\partial_\mu\partial_\nu + \xi(n) R^{(1)}(x) \right\}, \\ V^{(2)} \;&\equiv\; -\frac{1}{\sqrt{-g^{(0)}}} \left\{ [\partial_\mu \left(\sqrt{-g^{(0)}} \hat{h}^{\mu\nu}(x)\right)]\partial_\nu + \hat{h}^{\mu\nu}(x)\partial_\mu\partial_\nu - \xi(n) \left(R^{(2)}(x) + \frac{1}{2} h(x) R^{(1)}(x) \right) \right\}. \end{aligned} \tag{III.7}$$

---

[4]The Hartle-Hawking vacuum is a pure state which is perceived as vacuum by free-falling observers crossing either the black hole or the white hole horizons. However, if one traces out the modes localized in the second asymptotically flat region, one ends up with an incoherent density matrix perceived as a thermal state by static observers in the first asymptotically flat region.

In the above expressions, $R^{(k)}$ is the k-order term in the pertubation $h_{\mu\nu}(x)$ of the scalar curvature R and $\bar{h}_{\mu\nu}$ and $\hat{h}_{\mu\nu}$ denote a linear and a quadratic combination of the perturbation, respectively,

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h g^{(0)}_{\mu\nu}; \;\; \hat{h}_{\mu\nu} \equiv h_{\mu}^{\ \alpha} h_{\alpha\nu} - \frac{1}{2} h h_{\mu\nu} + \frac{1}{8} h^2 g^{(0)}_{\mu\nu} - \frac{1}{4} h_{\alpha\beta} h^{\alpha\beta} g^{(0)}_{\mu\nu}. \tag{III.8}$$

From quantum field theory in curved spacetime considerations we take the following action for the gravitational field (see [47,56] for more details)

$$S_g^{(div)}[g_{\mu\nu}] \;=\; \frac{1}{\ell_P^{n-2}} \int d^n x\, \sqrt{-g} R(x) + \frac{\alpha \bar{\mu}^{n-4}}{4(n-4)} \int d^n x\, \sqrt{-g} \left[3 R_{\mu\nu\alpha\beta}(x) R^{\mu\nu\alpha\beta}(x) \left(1 - 360 (\xi(n) - \frac{1}{6})^2 \right) R(x) R(x) \right]. \tag{III.9}$$

The first term is the classical Einstein-Hilbert action and the second term is the counterterm in four dimensions used to renormalize the divergent effective action. In this action $\ell_P^2 = 16\pi G$, $\alpha = (2880\pi^2)^{-1}$ and $\bar{\mu}$ is the renormalization mass scale.

We are interested in computing the CTP effective action for the model given by the form (III.6) for the matter action and when the field ϕ is initially in the Hartle- Hawking (HH) vacuum. Since the initial state of the field is described by a thermal density matrix at the HH temperature T_H[5], the finite temperature CTP effective action ($T \equiv 1/\beta$) for this model is given by (for details see [56])

$$\Gamma^{\beta}_{CTP}[h^{\pm}_{\mu\nu}] \;=\; S_g^{(div)}[h^{+}_{\mu\nu}] - S_g^{(div)}[h^{-}_{\mu\nu}] - \frac{i}{2} Tr\{\ln \bar{G}^{\beta}_{ab}[h^{\pm}_{\mu\nu}]\}, \tag{III.10}$$

where $\pm$ denote the forward and backward time path of the CTP formalism and $\bar{G}^{\beta}_{ab}[h^{\pm}_{\mu\nu}]$ is the complete 2×2 matrix propagator (a and b take $\pm$ values: G_{++}, G_{+-} and G_{--} correspond to the Feynman, Wightman and Schwinger Green's functions respectively) with thermal boundary conditions for the differential operator $\sqrt{-g^{(0)}}(\Box + V^{(1)} + V^{(2)} + \cdots)$. The actual form of $\bar{G}^{\beta}_{ab}$ cannot be explicitly given. However, it is easy to obtain a perturbative expansion in terms of $V^{(k)}_{ab}$, the k-order matrix version of the complete differential operator defined by $V^{(k)}_{\pm\pm} \equiv \pm V^{(k)}_{\pm}$ and $V^{(k)}_{\pm\mp} \equiv 0$, and G^{β}_{ab}, the thermal matrix propagator for a massless scalar field in Schwarzschild spacetime . To second order $\bar{G}^{\beta}_{ab}$ reads

$$\bar{G}^{\beta}_{ab} \;=\; G^{\beta}_{ab} - G^{\beta}_{ac} V^{(1)}_{cd} G^{\beta}_{db} - G^{\beta}_{ac} V^{(2)}_{cd} G^{\beta}_{db} + G^{\beta}_{ac} V^{(1)}_{cd} G^{\beta}_{de} V^{(1)}_{ef} G^{\beta}_{fb} + \cdots \tag{III.11}$$

Expanding the logarithm and dropping one term independent of the perturbation $h^{\pm}_{\mu\nu}(x)$, the CTP effective action may be perturbatively written as

$$\begin{aligned} \Gamma^{\beta}_{CTP}[h^{\pm}_{\mu\nu}] \;=\;& S_g^{div}[h^{+}_{\mu\nu}] - S_g^{div}[h^{-}_{\mu\nu}] + \frac{i}{2} Tr[V^{(1)}_{+} G^{\beta}_{++} - V^{(1)}_{-} G^{\beta}_{--} + V^{(2)}_{+} G^{\beta}_{++} - V^{(2)}_{-} G^{\beta}_{--}] \\ & - \frac{i}{4} Tr[V^{(1)}_{+} G^{\beta}_{++} V^{(1)}_{+} G^{\beta}_{++} + V^{(1)}_{-} G^{\beta}_{--} V^{(1)}_{-} G^{\beta}_{--} - 2 V^{(1)}_{+} G^{\beta}_{+-} V^{(1)}_{-} G^{\beta}_{-+}]. \end{aligned} \tag{III.12}$$

However, unlike the case of Ref. [56] where $h_{\mu\nu}$ represented a perturbation about flat space and hence one had knowledge of exact "unperturbed" thermal propagators, in this case, since the perturbation is about Schwarzschild spacetime, exact expressions for the corresponding unperturbed propagators $G^{\beta}_{ab}[h^{\pm}_{\mu\nu}]$ are not known. Therefore apart from the approximation of computing the CTP effective action to certain order

[5]As mentioned earlier, this is true for the basis of modes associated to static observers and provided that one is not concerned about quantum correlations (entanglement) between the two asymptotically flat regions, so that the modes localized in the second region can be traced out. Otherwise the full Hartle-Hawking vacuum, which is a pure state, should be considered.

in perturbation theory, an appropriate approximation scheme for the unperturbed Green's functions is also required. York used the Page approximation [57] for $\langle T_{\mu\nu}\rangle$ in the Schwarzschild metric. The additional complication here is that while to obtain $\langle T_{\mu\nu}\rangle$ the knowledge of only the thermal Feynman Green's function is required, to calculate the CTP effective action one needs the knowledge of the full matrix propagator, which involves the Feynman, Schwinger and Wightman functions. We can put aside the technical complexity in the calculation of the full thermal matrix propagator $G^{\beta}_{ab}[h^{\pm}_{\mu\nu}]$ as our main interest is to identify and analyze the noise term which is the new ingredient in the backreaction problem. We have mentioned that the noise term gives a stochastic contribution $T^{\rm s}_{\mu\nu}$ to the Einstein Langevin equation (II.2). We have also stated that this term is related to the variance of fluctuations in $T_{\mu\nu}$, i.e., schematically, to $\langle T^2_{\mu\nu}\rangle$. Since the Influence Functional or CTP formalism itself does not depend on the nature of the approximation, we will attempt to exhibit the general structure and project what is to be expected.

If we denote the difference and the sum of the perturbations $h^{\pm}_{\mu\nu}$ by $[h_{\mu\nu}] \equiv h^+_{\mu\nu} - h^-_{\mu\nu}$ and $\{h_{\mu\nu}\} \equiv h^+_{\mu\nu} + h^-_{\mu\nu}$, respectively, the influence functional form of the thermal CTP effective action may be written to second order in $h_{\mu\nu}$ as [56]

$$
\begin{aligned}
\Gamma^{\beta}_{CTP}[h^{\pm}_{\mu\nu}] \simeq & \frac{1}{2\ell_P^2}\int d^4x\ d^4x'\ [h_{\mu\nu}](x)L^{\mu\nu,\alpha\beta}_{(o)}(x,x')\{h_{\alpha\beta}\}(x') + \frac{1}{2}\int d^4x\ [h_{\mu\nu}](x)T^{\mu\nu}_{(\beta)}(x) \\
& +\frac{1}{2}\int d^4x\ d^4x'\ [h_{\mu\nu}](x)H^{\mu\nu,\alpha\beta}(x,x')\{h_{\alpha\beta}\}(x') - \frac{1}{2}\int d^4x\ d^4x'\ [h_{\mu\nu}](x)D^{\mu\nu,\alpha\beta}(x,x')\{h_{\alpha\beta}\}(x') \\
& +\frac{i}{2}\int d^4x\ d^4x'\ [h_{\mu\nu}](x)N^{\mu\nu,\alpha\beta}(x,x')[h_{\alpha\beta}](x'). \qquad \text{(III.13)}
\end{aligned}
$$

The first term is the Einstein-Hilbert action to second order in the perturbation $h^{\pm}_{\mu\nu}(x)$ and $\mathrm{L}^{\mu\nu,\alpha\beta}_{(o)}(x)$ is a symmetric local kernel, *i.e.* $\mathrm{L}^{\mu\nu,\alpha\beta}_{(o)}(x,x') = \mathrm{L}^{\mu\nu,\alpha\beta}_{(o)}(x',x)$. The second is a local term linear in $h^{\pm}_{\mu\nu}(x)$. $T^{\mu\nu}_{(\beta)}(x)$ represents the zeroth order contribution to $\langle \hat{T}_{\mu\nu}(x)\rangle$ and far away from the horizon it takes the form of the stress tensor of massless scalar particles at temperature β^{-1}. The third and fourth terms constitute the remaining quadratic component of the real part of the effective action. The kernels $H^{\mu\nu,\alpha\beta}(x,x')$ and $D^{\mu\nu,\alpha\beta}(x,x')$ are respectively even and odd in x, x'. The last term gives the imaginary part of the effective action and the kernel $N^{\mu\nu,\alpha\beta}(x,x')$ is symmetric. This is the general structure of the CTP effective action arising from the calculation of the traces in equation (III.12). Of course, to write down explicit expressions for the non-local kernels one requires the input of the explicit form of $G^{\beta}_{ab}[h^{\pm}_{\mu\nu}]$, which we have not used. In spite of this limitation we can make some interesting observations from this effective action. Connecting this thermal CTP effective action to the influence functional via equation (I.4) we see that the nonlocal imaginary term containing the kernel $N^{\mu\nu,\alpha\beta}(x,x')$ is responsible for the generation of the stochastic noise term in the Einstein-Langevin equation and the real non-local term containing kernel $D^{\mu\nu,\alpha\beta}(x,x')$ is responsible for the non-local dissipation term. The Einstein-Langevin equation can be generated from equation (I.7) by first constructing the improved semiclassical effective action in accordance with (I.6) and deriving the equation of motion (I.7) by taking a functional derivative of the above effective action with respect to $[h_{\mu\nu}]$ and equating it to zero. With the identification of the noise and dissipation kernels, one can use them to write down a Fluctuation-Dissipation relation (FDR) analogous to (I.9) in the context of black holes.

IV. STRUCTURE FORMATION IN INFLATIONARY UNIVERSES

Cosmological structure formation is a key problem in modern cosmology [59,58] and inflation [42–44] offers a natural solution to this problem. If an inflationary period is present, the initial seeds for the generation of the primordial inhomogeneities that lead to the large scale structure observed in the present universe have their source in the quantum fluctuations of the inflaton field, the field which is generally responsible for driving inflation. Stochastic gravity provides a sound and natural formalism for the derivation of the cosmological perturbations generated during inflation.

In Ref. [16] it was shown that the correlation functions that follow from the Einstein-Langevin equation which emerges in the framework of stochastic gravity coincide with that obtained with the usual quantization procedures [60] when both the metric perturbations and the inflaton fluctuations are both linearized. Stochastic gravity, however, can naturally deal with the fluctuations of the inflaton field even beyond the linear approximation.

Here we will illustrate the equivalence with the usual formalism, based on the quantization of the linear cosmological and inflaton perturbations, with one of the simplest chaotic inflationary models in which the background spacetime is a quasi de Sitter universe [15,16].

In this chaotic inflationary model [42,44] the inflaton field ϕ of mass m is described by the following Lagrangian density

$$\mathcal{L}(\phi) = \frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi + \frac{1}{2} m^2 \phi^2. \qquad \text{(IV.1)}$$

The conditions for the existence of an inflationary period, which is characterized by an accelerated cosmological expansion, is that the value of the field over a region with the typical size of the Hubble radius is higher than the Planck mass m_P. In order to solve the cosmological horizon and flatness problem more than 60 e-folds of expansion are needed; to achieve this the scalar field should begin with a value higher than $3m_P$. The inflaton mass is small: as we will see, the large scale anisotropies measured in the cosmic background radiation restrict the inflaton mass to be of the order of $10^{-6} m_P$. We will not discuss the naturalness of this inflationary model and we will simply assume that if one such region is found (inside a much larger universe) it will inflate to become our observable universe.

We want to study the metric perturbations produced by the stress-energy tensor fluctuations of the inflaton field on the homogeneous background of a flat Friedmann-Robertson-Walker model, described by the cosmological scale factor $a(\eta)$, where η is the conformal time, which is driven by the homogeneous inflaton field $\phi(\eta) = \langle \hat{\phi} \rangle$. Thus we write the inflaton field in the following form: $\hat{\phi} = \phi(\eta) + \hat{\varphi}(x)$, where $\hat{\varphi}(x)$ corresponds to a free massive quantum scalar field with zero expectation value on the homogeneous background metric: $\langle \hat{\varphi} \rangle = 0$. We will restrict ourselves to scalar-type metric perturbations because these are the ones that couple to the inflaton fluctuations in the linear theory. We note that this is not so if we were to consider inflaton fluctuations beyond the linear approximation, then tensorial and vectorial metric perturbations would also be driven. The perturbed metric $\tilde{g}_{ab} = g_{ab} + h_{ab}$ can be written in the longitudinal gauge as,

$$ds^2 = a^2(\eta)[-(1 + 2\Phi(x)) d\eta^2 + (1 - 2\Psi(x)) \delta_{ij} dx^i dx^j], \qquad \text{(IV.2)}$$

where the scalar metric perturbations $\Phi(x)$ and $\Psi(x)$ correspond to Bardeen's gauge invariant variables [61].

A. Einstein-Langevin equation for scalar metric perturbations

The Einstein-Langevin equation is gauge invariant, thus we can work in a desired gauge and then extract the gauge invariant quantities. It is given by

$$G^{(0)}_{ab} - 8\pi G \langle \hat{T}^{(0)}_{ab} \rangle + G^{(1)}_{ab}(h) - 8\pi G \langle \hat{T}^{(1)}_{ab}(h) \rangle = 8\pi G \xi_{ab}, \qquad \text{(IV.3)}$$

where the two first terms cancel, that is $G^{(0)}_{ab} - 8\pi G \langle \hat{T}^{(0)}_{ab} \rangle = 0$, as the background metric satisfies the semiclassical Einstein equations. Here the subscripts (0) and (1) refer to functions in the background metric g_{ab} and linear in the metric perturbation h_{ab}, respectively. The stress tensor operator $\hat{T}_{ab}$ for the minimally coupled inflaton field in the perturbed metric is:

$$\hat{T}_{ab} = \tilde{\nabla}_a \hat{\phi} \tilde{\nabla}_b \hat{\phi} + \frac{1}{2} \tilde{g}_{ab} (\tilde{\nabla}_c \hat{\phi} \tilde{\nabla}^c \hat{\phi} + m^2 \hat{\phi}^2). \qquad \text{(IV.4)}$$

Using the decomposition of the scalar field into its homogeneous and inhomogeneous part and the metric $\tilde{g}_{ab}$ into its homogeneous background g_{ab} and its perturbation h_{ab}, the renormalized expectation value for the stress-energy tensor operator can be written as

$$\langle \hat{T}^R_{ab}[\tilde{g}]\rangle = \langle \hat{T}_{ab}[\tilde{g}]\rangle_{\phi\phi} + \langle \hat{T}_{ab}[\tilde{g}]\rangle_{\phi\varphi} + \langle \hat{T}^R_{ab}[\tilde{g}]\rangle_{\varphi\varphi}, \tag{IV.5}$$

where the subindices indicate the degree of dependence on the homogeneous field ϕ and its perturbation φ. The first term in this equation depends only on the homogeneous field and it is given by the classical expression. The second term is proportional to $\langle\hat{\varphi}[\tilde{g}]\rangle$ which is not zero because the field dynamics is considered on the perturbed spacetime, *i.e.*, this term includes the coupling of the field with h_{ab} and may be obtained from the expectation value of the linearized Klein-Gordon equation, $\left(\Box_{g+h} - m^2\right)\hat{\varphi} = 0$. The last term in Eq. (IV.5) corresponds to the expectation value to the stress tensor for a free scalar field on the spacetime of the perturbed metric.

After using the previous decomposition, the noise kernel $N_{abcd}[g;x,y)$ can be written as

$$\langle\{\hat{t}_{ab}[g;x),\hat{t}_{cd}[g;y)\}\rangle \;=\; \langle\{\hat{t}_{ab}[g;x),\hat{t}_{cd}[g;y)\}\rangle_{(\phi\varphi)^2} + \langle\{\hat{t}_{ab}[g;x),\hat{t}_{cd}[g;y)\}\rangle_{(\varphi\varphi)^2}, \tag{IV.6}$$

where we have used the fact that $\langle\hat{\varphi}\rangle = 0 = \langle\hat{\varphi}\hat{\varphi}\hat{\varphi}\rangle$ for Gaussian states on the background geometry. We consider the vacuum state to be the Euclidean vacuum which is preferred in the de Sitter background, and this state is Gaussian. In the above equation the first term is quadratic in $\hat{\varphi}$ whereas the second one is quartic, both contributions to the noise kernel are separately conserved since both $\phi(\eta)$ and $\hat{\varphi}$ satisfy the Klein-Gordon field equations on the background spacetime. Consequently, the two terms can be considered separately. On the other hand if one treats $\hat{\varphi}$ as a small perturbation the second term in (IV.6) is of lower order than the first and may be consistently neglected, this corresponds to neglecting the last term of Eq. (IV.5). The stress tensor fluctuations due to a term of that kind were considered in Ref. [15].

We can now write down the Einstein-Langevin equations (IV.3) to linear order in the inflaton fluctuations. It is easy to check [16] that the *space-space* components coming from the stress tensor expectation value terms and the stochastic tensor are diagonal, i.e. $\langle\hat{T}_{ij}\rangle = 0 = \xi_{ij}$ for $i \neq j$. This, in turn, implies that the two functions characterizing the scalar metric perturbations are equal: $\Phi = \Psi$ in agreement with Ref. [60]. The equation for Φ can be obtained from the $0i$-component of the Einstein-Langevin equation, which in Fourier space reads

$$2ik_i(\mathcal{H}\Phi_k + \Phi'_k) = 8\pi G(\xi_{0i})_k, \tag{IV.7}$$

where k_i is the comoving momentum component associated to the comoving coordinate x^i, and we have used the definition $\Phi_k(\eta) = \int d^3x \exp(-i\vec{k}\cdot\vec{x})\Phi(\eta,\vec{x})$. Here primes denote derivatives with respect to the conformal time η and $\mathcal{H} = a'/a$. A nonlocal term of dissipative character which comes from the second term in Eq. (IV.5) should also appear on the left hand side of Eq. (IV.7), but we have neglected it to simplify the forthcoming expressions[6]. Note, however, that the equivalence of the stochastic approach to linear order in $\hat{\varphi}$ and the usual linear cosmological perturbations approach is independent of that approximation [16]. To solve Eq. (IV.7), whose left-hand side comes from the linearized Einstein tensor for the perturbed metric [60], we need the retarded propagator for the gravitational potential Φ_k,

$$G_k(\eta,\eta') = -i\frac{4\pi}{k_i m_P^2}\left(\theta(\eta-\eta')\frac{a(\eta')}{a(\eta)} + f(\eta,\eta')\right), \tag{IV.8}$$

where f is a homogeneous solution of Eq. (IV.7) related to the initial conditions chosen and $m_P^2 = 1/G$. For instance, if we take $f(\eta,\eta') = -\theta(\eta_0-\eta')a(\eta')/a(\eta)$ the solution would correspond to "turning on" the stochastic source at η_0. With the solution of the Einstein-Langevin equation (IV.7) for the scalar metric perturbations we are in a position to compute the two-point correlation functions for these perturbations.

---

[6]Such a term, which leads to an integrodifferential Einstein-Langevin equation, might not be negligible in some situations. Despite the apparent difficulty of dealing with an integrodifferntial equation, in this case the Einstein-Langevin equation can be actually transformed, after suitable manipulation, into an ordinary differential equation, so that the inclusion of the nonlocal term is still tractable [16]

B. Correlation functions for scalar metric perturbations

The two-point correlation function for the scalar metric perturbations induced by the inflaton fluctuations is thus given by

$$\langle \Phi_k(\eta)\Phi_{k'}(\eta')\rangle_s = (2\pi)^2\delta(\vec{k}+\vec{k}') \times \int^{\eta} d\eta_1 \int^{\eta'} d\eta_2 G_k(\eta,\eta_1)G_{k'}(\eta',\eta_2)\langle(\xi_{0i})_k(\eta_1)(\xi_{0i})_{k'}(\eta_2)\rangle_s. \qquad \text{(IV.9)}$$

Here the two-point correlation function for the stochastic source, which is connected to the stress-energy tensor fluctuations through the noise kernel is given by,

$$\langle(\xi_{0i})_k(\eta_1)(\xi_{0i})_{-k}(\eta_2)\rangle_s \;=\; \frac{1}{2}\langle\{(\hat{t}_{0i})_k(\eta_1),(\hat{t}_{0i})_{-k}(\eta_2)\}\rangle_{\phi\varphi} \;=\; \frac{1}{2}k_i k_i \phi'(\eta_1)\phi'(\eta_2)G_k^{(1)}(\eta_1,\eta_2), \qquad \text{(IV.10)}$$

where $G_k^{(1)}(\eta_1,\eta_2) = \langle\{\hat{\varphi}_k(\eta_1),\hat{\varphi}_{-k}(\eta_2)\}\rangle$ is the k-mode Hadamard function for a free minimally coupled scalar field in the appropriate vacuum state on the Friedmann-Robertson-Walker background.

In practice, to make the explicit computation of the Hadamard function we will assume that the field state is in the Euclidean vacuum and the background spacetime is de Sitter. Furthermore, we will compute the Hadamard function for a massless field, and will make a perturbative expansion in terms of the dimensionless parameter m/m_P. Thus we consider

$$\bar{G}_k^{(1)}(\eta_1,\eta_2) = \langle 0|\{\hat{y}_k(\eta_1),\hat{y}_{-k}(\eta_2)\}|0\rangle = 2\mathcal{R}\left(u_k(\eta_1)u_k^*(\eta_2)\right),$$

with $\hat{y}_k(\eta) = a(\eta)\hat{\varphi}_k(\eta) = \hat{a}_k u_k(\eta) + \hat{a}^\dagger_{-k}u^*_{-k}(\eta)$ and where $u_k = (2k)^{-1/2}e^{ik\eta}(1-i/\eta)$ are the positive frequency k-mode for a massless minimally coupled scalar field on a de Sitter background, which define the Euclidean vacuum state: $\hat{a}_k|0\rangle = 0$ [31].

The assumption of a massless field for the computation of the Hadamard function is made because massless modes in de Sitter are much simpler to deal with than massive modes. We can see that this is, however, a reasonable approximation as follows. For a given mode the $m = 0$ approximation is reasonable when its wavelength λ is shorter that the Compton wavelength, $\lambda_c = 1/m$. In our case we have a very small mass m and the horizon size H^{-1}, where H is the Hubble constant $H = \dot{a}/a$ (here $a(t)$ with t the physical time $dt = ad\eta$) satisfies that $H^{-1} < \lambda_c$. Thus, for modes inside the horizon $\lambda < \lambda_c$ and $m = 0$ is a reasonable approximation. Outside the horizon massive modes decay in amplitude as $\sim \exp(-m^2t/H)$ whereas massless modes remain constant, thus when modes leave the horizon the approximation will eventually break down. However, we only need to ensure that the approximation is still valid after 60 e-folds, *i.e.* $Ht \sim 60$, but this is the case since $60\, m^2 < H^2$ given that $m \sim 10^{-6}m_P$, and $m \ll H$ as in most inflationary models [43,59].

The background geometry is not exactly that of de Sitter spacetime, for which $a(\eta) = -(H\eta)^{-1}$ with $-\infty < \eta < 0$. One can expand in terms of the "slow-roll" parameters and assume that to first order $\dot{\phi}(t) \simeq m_P^2(m/m_P)$, where t is the physical time. The correlation function for the metric perturbation (IV.9) can then be easily computed; see Ref. [15,16] for details. The final result, however, is very weakly dependent on the initial conditions as one may understand from the fact that the accelerated expansion of de quasi-de Sitter spacetime during inflation erases the information about the initial conditions. Thus one may take the initial time to be $\eta_0 = -\infty$ and obtain to lowest order in m/m_P the expression

$$\langle \Phi_k(\eta)\Phi_{k'}(\eta')\rangle_s \simeq 8\pi^2(m/m_P)^2k^{-3}(2\pi)^3\delta(\vec{k}+\vec{k}')\cos k(\eta-\eta'). \qquad \text{(IV.11)}$$

From this result two main conclusions are derived. First, the prediction of an almost Harrison-Zel'dovich scale-invariant spectrum for large scales, i.e. small values of k. Second, since the correlation function is of order of $(m/m_P)^2$ a severe bound to the mass m is imposed by the gravitational fluctuations derived from the small values of the Cosmic Microwave Background (CMB) anisotropies detected by COBE [60]. This bound is of the order of $(m/m_P) \sim 10^{-6}$.

We should now comment on some differences with those works in Ref. [63] which used a self-interacting scalar field or a scalar field interacting nonlinearly with other fields. In those works an important relaxation of the ratio m/m_p was found. The long wavelength modes of the inflaton field were regarded as an open

system in an environment made out of the shorter wavelength modes. Then, Langevin type equations were used to compute the correlations of the long wavelength modes driven by the fluctuations of the shorter wavelength modes. In order to get a significant relaxation on the above ratio, however, one had to assume that the correlations of the free long wavelength modes, which correspond to the dispersion of the system initial state, had to be very small. Otherwise they dominate by several orders of magnitude those fluctuations that come from the noise of the environment. This would require a great amount of fine-tuning for the initial quantum state of each mode [16]. We should remark that in the model discussed here there is no environment for the inflaton fluctuations. The inflaton fluctuations, however, are responsible for the noise that induce the metric perturbations.

Acknowledgements This work is supported in part by NSF grant PHY98-00967, the MICYT Research Project No. FPA-2001-3598 and European project HPRN-CT-2000-00131.

---

[1] B. L. Hu and A. Matacz, "Quantum Noise in Gravitation and Cosmology" in *Fluctuations and Order*, edited by Marko Millonas (Springer-Verlag, Berlin, 1996). [astro-ph/9312012]
[2] B. L. Hu, in *Proceedings of the Third International Workshop on Thermal Fields and its Applications*, edited by R. Kobes and G. Kunstatter (World Scientific, Singapore, 1994)[gr-qc/9403061]
[3] E. Calzetta, B.-L. Hu, F. D. Mazzitelli, Phys. Rep. **352**, 459-520 (2001).
[4] S. A. Ramsey and B. L. Hu, Phys. Rev. D **56**, 678 (1997).
[5] S. A. Ramsey, B. L. Hu and A. M. Stylianopoulos, Phys. Rev. D **57**, 6003 (1998)
[6] A. Raval, B.L. Hu, and J. Anglin, Phys. Rev. D **53**, 7003-7019 (1996).
[7] A. Raval, B. L. Hu and D. Koks, Phys. Rev. **D 55**, 4795 (1997).
[8] D. Koks, B. L. Hu, A. Matacz, and A. Raval, Phys. Rev. D **56**, 4905-4915 (1997).
[9] D. Koks, A. Matacz, and B. L. Hu, Phys. Rev. D **55**, 5917-5935 (1997)
[10] E. Calzetta and B. L. Hu, Phys. Rev. D **61**, 025012 (2000).
[11] E. Calzetta and B. L. Hu, in preparation.
[12] G. J. Stephens, E. A. Calzetta, B. L. Hu, and S. A. Ramsey, Phys. Rev. D **59**, 045009 (1999)
[13] E. Calzetta, A. Roura and E. Verdaguer, Physica A **319**, 188 (2003).
[14] E. Calzetta, A. Roura and E. Verdaguer, Phys. Rev. D **64**, 105008 (2001); Phys. Rev. Lett. **88**, 010403 (2002)
[15] A. Roura and E. Verdaguer, Int. J. Theor. Phys. **39**, 1831 (2000).
[16] A. Roura and E. Verdaguer, in preparation.
[17] B. L. Hu and K. Shiokawa, Phys. Rev. D **57**, 3474-3483 (1998)
[18] K. Shiokawa, Phys. Rev. D **62**, 024002 (2000).
[19] B. L. Hu, A. Raval and S. Sinha, "Notes on Black Hole Fluctuations and Backreaction" in *Black Holes, Gravitational Radiation and the Universe: Essays in honor of C. V. Vishveshwara* eds. B. Iyer and B. Bhawal (Kluwer Academic Publishers, Dordrecht, 1998) [gr-qc/9901010].
[20] S. Sinha, Alpan Raval and B. L. Hu, "Black Hole Fluctuations and Backreaction in Stochastic Gravity", Found. Phys. 33 (2003) 37-64 [gr-qc/0210013].
[21] G. Stephens and B. L. Hu, "Notes on Black Hole Phase Transitions", Int. J. Theor. Phys. 40 (2001) 2183-2200 [gr-qc/0102052]
[22] D. J. Gross, M. J. Perry and L. G. Yaffe, Phys. Rev. D **25**, 330 (1982).
[23] B. F. Whiting and J. W. York, Jr., Phys. Rev. Lett. 61, 1336 (1988)
[24] B. L. Hu, "Stochastic Gravity", Int. J. Theor. Phys. **38**, 2987 (1999) [gr-qc/9902064].
[25] B. L. Hu, E. Verdaguer, "Recent Advances in Stochastic Gravity: Theory and Issues" Erice Lectures May 2001, in *Advances in the Interplay between Quantum and Gravity Physics* edited by P. Bergmann and V. De Sabbata, (Kluwer, Dortrecht, 2002) [gr-qc/0110092].
[26] B. L. Hu and E. Verdaguer, "Stochastic Gravity: A Primer with Applications", Class. Quant. Grav. 20 (2003) R1-R42
[27] B. L. Hu and E. Verdaguer, "Stochastic Gravity: Theory and Applications", *Living Reviews in Relativity* (2003)
[28] A. Roura and E. Verdaguer, in preparation.

[29] B. L. Hu, "A Kinetic Theory Approach to Quantum Gravity", Int. J. Theor. Phys. (2002) [gr-qc/0204069]
[30] B. S. DeWitt, Phys. Rep. 19, 295(1975).
[31] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).
[32] E. Calzetta and B. L. Hu, Phys. Rev. D **35**, 495 (1987); A. Campos and E. Verdaguer, Phys. Rev. **D49**, 1861 (1994).
[33] B. L. Hu, "Semiclassical Gravity and Mesoscopic Physics" in *Quantum Classical Correspondence* eds. D. S. Feng and B. L. Hu (International Press, Boston, 1997) [gr-qc/9511077].
[34] B. L. Hu, "General Relativity as Geometro-Hydrodynamics" Invited talk at the Second Sakharov Conference, Moscow, May, 1996 [gr-qc/9607070].
[35] R. Feynman and F. Vernon, Ann. Phys. (NY) **24**, 118 (1963); R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals*, (McGraw - Hill, New York, 1965).
[36] A. O. Caldeira and A. J. Leggett, Physica **121A**, 587 (1983); Ann. Phys. (NY) **149**, 374 (1983). H. Grabert, P. Schramm and G. L. Ingold, Phys. Rep. **168**, 115 (1988). B. L. Hu, J. P. Paz and Y. Zhang, Phys. Rev. **D45**, 2843 (1992); **D47**, 1576 (1993).
[37] B. L. Hu and A. Matacz, Phys. Rev. D49, 6612(1994)
[38] J. Schwinger, J. Math. Phys. **2** (1961) 407; L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47** , 1515 (1964) [Engl. trans. Sov. Phys. JEPT **20**, 1018 (1965)] R. D. Jordan, Phys. Rev. **D33** , 444 (1986); E. Calzetta and B. L. Hu, Phys. Rev. D **35**, 495 (1987);**37**, 2878 (1988); **D40**, 656 (1989).
[39] E. Calzetta and B. L. Hu, Phys. Rev. D49, 6636(1994). B. L. Hu and A. Matacz, Phys. Rev. **D51**, 1577 (1995). B. L. Hu and S. Sinha, Phys. Rev. **D51**, 1587 (1995). A. Campos and E. Verdaguer, Phys. Rev. D **53**, 1927 (1996).
[40] L. Parker, Phys. Rev. **183**, 1057 (1969).
[41] S. W. Hawking, Comm. Math. Phys. **43**, 199 (1975).
[42] A. H. Guth, Phys. Rev. D 23, 347 (1981). A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982). A. D. Linde, Phys. Lett. 114B, 431 (1982). Phys. Lett. 162B, 281 (1985)
[43] E.W. Kolb and M. Turner, *The early Universe* (Addison-Wesley, Reading, Massachusetts, 1990); T. Padmanabhan, *Structure formation* (Cambridge University Press, Cambridge, England, 1993).
[44] A. Linde, *Particle physics and inflationary cosmology* (Harwood Academic Publishers, Switzerland, 1990).
[45] B. L. Hu, Physica A158, 399 (1979).
[46] R. Martin and E. Verdaguer, Phys. Lett. B **465**, 113 (1999).
[47] R. Martin and E. Verdaguer, Phys. Rev. D **60**, 084008 (1999).
[48] R. Martin and E. Verdaguer, Phys. Rev. D **61**, 124024 (2000).
[49] N. G. Phillips and B. L. Hu, Phys. Rev. D **55**, 6132 (1997).
[50] B. L. Hu and N. G. Phillips, Int. J. Theor. Phys. **39**, 1817 (2000) [gr-qc/0004006].
[51] N. G. Phillips and B. L. Hu, Phys. Rev. D **62**, 084017 (2000).
[52] N. G. Phillips and B. L. Hu, Phys. Rev. D **63**, 104001 (2001).
[53] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[54] J. M. Bardeen, Phys. Rev. Lett. 46, 382 (1981); P. Hajicek and W. Israel, Phys. Lett. 80A, 9 (1980).
[55] J. W. York, Jr., Phys. Rev. D **28**, 2929 (1983); D **31**, 775 (1985); D **33**, 2092 (1986).
[56] A. Campos and B. L. Hu, Phys. Rev. D58, 125021(1998); Int. J. Theor. Phys. 38, 1253(1999)
[57] D. N. Page, Phys. Rev. **D25**, 1499 (1982).
[58] G.F. Smoot *et al.*, Astrophys. J. Lett. **396**, L1(1992).
[59] T. Padmanabhan, *Structure Formation* (Cambridge University Press, Cambridge, England, 1993).
[60] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, Phys. Rep. **215**, 203 (1992).
[61] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[62] A. Roura and E. Verdaguer, Int. J. Theor. Phys. **38**, 3123 (1999).
[63] E. Calzetta and B. L. Hu, Phys. Rev. D **52**, 6770 (1995); A. Matacz, Phys. Rev. D **55**, 1860 (1997); **56**, 1836 (1997); E. Calzetta and S. Gonorazky, Phys. Rev. D **55**, 1812 (1997)